札幌大学総合論叢　第24号（2007年10月）

〈研究ノート〉

Efficient Equilibrium Contracts in Two-Player Games*

Yamada Akira

Abstract

We analyze two-player two-stage games where players may make binding offers of schemes for side payment acceptance (or rejection) as well as those for side payments before choosing actions. Each agent's receipt scheme depends on the other's transfer scheme, but each agent's transfer scheme does not depend on the other's receipt scheme. We find that any set of efficient actions maximizing the total payoff is played on an equilibrium path of the two-stage game even with such partially interdependent bilateral side contracts, when players are only two and there is a pure Nash equilibrium in the underlying game (the second stage game without side contracts).

1. Introduction

Coase (1960) put forth an idea that if property rights are well-defined, and bargaining is costless, then rational agents playing a game with externalities should contract to reach an efficient point. However, Jackson and Wilkie (2005) explored two-stage games where agents may make binding offers of strategy-contingent side payments before choosing actions, and found that if there are only two agents, the agents are not always able to come to an agreement that supports an efficient strategy profile as an equilibrium point of the game, even if there are no transactions costs and complete information, and moreover there is a pure Nash equilibrium in the underlying game (the second stage game without side

* This paper was written with the support of a 2006 academic research grant provided by Sapporo University.

contracts).

Yamada (2003) analyzed two-stage games where players may make binding offers of schemes for side payment acceptance (or rejection) as well as those for side payments before choosing actions. A side payment from a player, say 1, to another, say 2, is implemented if and only if 1 offers the payment and 2 accepts it. If 2 rejects, then 1's offer is not in effect, and the payoff for the transfer remains with 1. As a result, Yamada (2003) specified a class of bilateral side contracts which may induce play of efficient actions in equilibria no matter what number of players there are. However, the contracts proposed by Yamada (2003) are fully interdependent and considerably complicated: not only does each agent's receipt scheme depend on the others' transfer schemes, but also each agent's transfer scheme indirectly depends on the others' receipt schemes.

Is there any class of simpler side contracts to lead to efficiency generally? This is the question we are to address next. In the present note we will study two-player two-stage games where players may make binding offers of schemes for side payment acceptance (or rejection) as well as those for side payments before choosing actions, like in Yamada (2003), but the bilateral side contracts (transfer and receipt schemes) are only partially interdependent: each agent's receipt scheme depends on the other's transfer scheme while not vice versa, in contrast to Yamada (2003). Then we will see that even with such simpler side contracts every efficient strategy profile is played on an equilibrium path of the two-stage game if there is a pure Nash equilibrium in the underlying game. In addition, we will reach a similar result even when equilibrium contracts are required to meet agents' budget constraint with their transfer.

In what follows we present the model in Section 2 and our analysis in Section 3. Our concluding remarks appear in Section 4.

2. The Model

We consider two-stage games played as follows.

Stage 1: Each player announces a transfer function (transfer scheme) and a receipt function (transfer acceptance/rejection scheme), both of which are assumed to be binding.[1]

Stage 2: Each player chooses an action.

2. 1 The underlying game

There are two players 1 and 2. A player i's finite pure strategy space in the second stage game is denoted by X_i, with $X = X_1 \times X_2$. Let $\Delta(X_i)$ denote the set of mixed strategies for i, and let $\Delta = \Delta(X_1) \times \Delta(X_2)$. We denote by x_i, x, μ_i, and μ generic elements of X_i, X, $\Delta(X_i)$, and Δ respectively. For simplicity, we sometimes use x_i and x to denote μ_i and μ respectively that place probability one on x_i and x. A player i's payoffs in the second stage game are given by a von Neumann-Morgenstern utility function $v_i: X \rightarrow \mathbb{R}$. Let $v = (v_1, v_2)$.

2. 2 The contracts

We are interested in the contracts that are partially interdependent: each agent's receipt scheme depends on the other's transfer schemes.

A transfer function announced by player i in the first stage is denoted by t_i, where $t_i : X \rightarrow \mathbb{R}_+$ represents i's promises to the other j as a function of actions chosen in the second stage. Let T be the set of all possible t_i. Let $t = (t_1, t_2)$. A transfer function t_i announced by player i meets his budget constraint if $t_i(x) \leqq \max\{0, v_i(x)\}$ for all x. A pair $t = (t_1, t_2)$ of transfer functions is called feasible if both t_1 and t_2 meet their budget constraint.

A receipt function announced by player i in the first stage is denoted by r_i, where $r_i : T \rightarrow \{0, 1\}$ represents i's acceptance (1) or rejection (0) of transfers from the other j as a function of transfer functions announced by j in the first stage. Let $r = (r_1, r_2)$.

Given a pair t of transfer functions and a pair r of receipt functions in the first stage, and a play x in the second stage game, the payoff U_i to player i becomes

$$U_i(x, t, r) = v_i(x) + (r_i(t_j)t_j(x) - r_j(t_i)t_i(x)).$$

Given a pair t of transfer functions and a pair r of receipt functions in the first stage, and a play μ in the second stage game, the expected payoff EU_i to player i becomes

$$EU_i(\mu, t, r) = \sum_x \mu_1(x_1)\mu_2(x_2)(v_i(x) + (r_i(t_j)t_j(x) - r_j(t_i)t_i(x))).$$

[1] Yamada (2005) characterized efficient equilibrium outcomes of two-player games that remain equilibrium outcomes even when the two players may *alternately* make binding offers of strategy contingent side payments before the game is played.

Let $NE(t, r)$ denote the set of (mixed) Nash equilibria of the second stage game given (t, r) in the first stage. Let NE represent the set of (mixed) Nash equilibria of the underlying game (the second stage game without side contracts).

A pure strategy profile $x \in X$ of the second stage game together with a vector $u \in \mathbb{R}^2$ of payoffs such that $\sum_i u_i = \sum_i v_i(x)$ is *supportable* if there exists a subgame perfect equilibrium of the two-stage game where some t and some r are announced in the first stage and x is played in the second stage on the equilibrium path, and $U_i(x, t, r) = u_i$.

A pure strategy profile $x \in X$ of the second stage game together with a vector $u \in \mathbb{R}^2$ of payoffs such that $\sum_i u_i = \sum_i v_i(x)$ is *feasibly supportable* if there exists a subgame perfect equilibrium of the two-stage game where some feasible t and some r are announced in the first stage and x is played in the second stage on the equilibrium path, and $U_i(x, t, r) = u_i$.

3. Analysis

The following proposition holds in the model, which implies that any set of efficient actions maximizing the total payoff is supportable with some payoff distribution when there is a pure Nash equilibrium in the underlying game.

Proposition 1. *$(\bar{x}, \bar{u})$ such that $\sum_i \bar{u}_i = \sum_i v_i(\bar{x}) \geqq \sum_i v_i(x)$ for all $x \in X$ is supportable if there exists ${}_i x$ for all i such that ${}_i x \in NE$ and $v_i({}_i x) \leqq \bar{u}_i$.*

Proof of Proposition 1. Suppose for $(\bar{x}, \bar{u})$ with $\sum_i \bar{u}_i = \sum_i v_i(\bar{x}) \geqq \sum_i v_i(x)$ for all $x \in X$, there exists ${}_i x$ for all i such that ${}_i x \in NE$ and $v_i({}_i x) \leqq \bar{u}_i$. Let $\bar{t}$ and $\bar{r}$ be as follows.

$$\bar{t}_i(x) = \begin{cases} \max\{0, v_i(x) - \bar{u}_i\} & \text{if } x_j = \bar{x}_j \\ 0 & \text{otherwise.} \end{cases}$$

$$\bar{r}_i(t_j) = \begin{cases} 1 & \text{if } t_j = \bar{t}_j \\ 0 & \text{otherwise.} \end{cases}$$

Note $\bar{x} \in NE(\bar{t}, \bar{r})$ and $U_i(\bar{x}, \bar{t}, \bar{r}) = \bar{u}_i$.

Consider the following strategy profile (μ, t, r).

(1) $(t, r) = (\bar{t}, \bar{r})$;

(2) if (t, r) and $(\bar{t}, (\bar{r}_j, r_i))$ and $r_i(\bar{t}_j) = 1$, then $\mu = \bar{x}$;

(2-1) if $(t, r) = (\bar{t}, (\bar{r}_j, r_i))$ and $r_i(\bar{t}_j) = 0$, then $\mu \in \{{}_ix, ({}_ix_i, \bar{x}_j)\} \cap NE(t, r)$;

(2-2) if $(t, r) = ((\bar{t}_j\, t_i), (\bar{r}_j, r_i))$, $t_i \neq \bar{t}_i$, and $r_i(\bar{t}_j) = 0$, then $\mu = {}_ix$;

(2-3) if $(t, r) = ((\bar{t}_j\, t_i), (\bar{r}_j, r_i))$, $t_i \neq \bar{t}_i$, and $r_i(\bar{t}_j) = 1$, then $\mu \in \{{}_ix, ({}_ix_j, \bar{x}_i)\} \cap NE(t, r)$;

(2-4) otherwise $\mu \in NE\ (t, r)$.

Suppose $(t, r) = (\bar{t}, (\bar{r}_j, r_i))$ and $r_i(\bar{t}_j) = 1$ for some i. Then, $NE(t, r) = NE(\bar{t}, \bar{r})$. Hence $\bar{x} \in NE(t, r)$, and $U_i(\bar{x}, t, r) = U_i(\bar{x}, \bar{t}, \bar{r}) = \bar{u}_i$ in the subgame (2).

Suppose $(t, r) = (\bar{t}, (\bar{r}_j, r_i))$ and $r_i(\bar{t}_j) = 0$ for some i. If $\mu = x = ({}_ix_i, x_j)$, then

$$
\begin{aligned}
U_j(x, t, r) &= v_j(x) + (\bar{r}_j(\bar{t}_i)\bar{t}_i(x) - r_i(\bar{t}_j)\bar{t}_j(x)) \\
&= v_j(x) + (1 \cdot \bar{t}_i(x) - 0 \cdot \bar{t}_j(x)) \\
&= v_j(x) + \bar{t}_i(x) \\
&= \begin{cases} v_j(x) + \max\{0, v_i(x) - \bar{u}_i\} & \text{if } x_j = \bar{x}_j \\ v_j(x) & \text{otherwise} \end{cases}
\end{aligned}
$$

while if $\mu = x = ({}_ix_j, x_i)$, then

$$
\begin{aligned}
U_i(x, t, r) &= v_i(x) + (r_i(\bar{t}_j)\bar{t}_j(x) - \bar{r}_j(\bar{t}_i)\bar{t}_i(x)) \\
&= v_i(x) + (0 \cdot \bar{t}_j(x) - 1 \cdot \bar{t}_i(x)) \\
&= v_i(x) - \bar{t}_i(x) \\
&\leqq v_i(x) \leqq v_i({}_ix).
\end{aligned}
$$

Hence, if ${}_ix \notin NE(t, r)$, then $\bar{t}_i({}_ix_i, \bar{x}_j) > 0$ and $U_j(({}_ix_i, \bar{x}_j), t, r) \geqq U_j(({}_ix_i, x'_j), t, r)$ for all x'_j. Moreover, $\bar{t}_i({}_ix_i, \bar{x}_j) > 0$ implies $U_i(({}_ix_i, \bar{x}_j), t, r) = \bar{u}_i \geqq U_i((x'_i, \bar{x}_j), t, r)$ for all x'_i. That is, if ${}_ix \notin NE(t, r)$, then $({}_ix_i, \bar{x}_j) \in NE(t, r)$. Thus, $\{{}_ix, ({}_ix_i, \bar{x}_j)\} \cap NE(t, r) \neq \emptyset$, and $U_i({}_ix, t, r) = v_i({}_ix) - \bar{t}_i({}_ix) = v_i({}_ix) \leqq \bar{u}_i$ while $U_i(({}_ix_i, \bar{x}_j), t, r) = v_i({}_ix_i, \bar{x}_j) - \max\{0, v_i({}_ix_i, \bar{x}_j) - \bar{u}_i\} \leqq \bar{u}_i$ in the subgame (2-1).

Suppose $(t, r) = ((\bar{t}_j, t_i), (\bar{r}_j, r_i))$, $t_i \neq \bar{t}_i$, and $r_i(\bar{t}_j) = 0$. Then, $NE(t, r) = NE$ since $\bar{r}_j(t_i) = 0$ as well. Hence ${}_ix \in NE(t, r)$, and $U_i({}_ix, t, r) = v_i({}_ix) \leqq \bar{u}_i$ in the subgame (2-2).

Suppose $(t, r) = ((\bar{t}_j, t_i), (\bar{r}_j, r_i))$, $t_i \neq \bar{t}_i$, and $r_i(\bar{t}_j) = 1$. If $\mu = x = ({}_ix_j, x_i)$, then

$$
\begin{aligned}
U_i(x, t, r) &= v_i(x) + (r_i(\bar{t}_j)\bar{t}_j(x) - \bar{r}_j(t_i)t_i(x)) \\
&= v_i(x) + (1 \cdot \bar{t}_j(x) - 0 \cdot t_i(x)) \\
&= v_i(x) + \bar{t}_j(x) \\
&= \begin{cases} v_i(x) + \max\{0, v_j(x) - \bar{u}_j\} & \text{if } x_i = \bar{x}_i \\ v_i(x) & \text{otherwise} \end{cases}
\end{aligned}
$$

while if $\mu = x = ({}_i x_i, x_j)$, then

$$U_j(x, t, r) = v_j(x) + (\bar{r}_j(t_i)t_i(x) - r_i(\bar{t}_j)\bar{t}_j(x))$$
$$= v_j(x) + (0 \cdot t_i(x) - 1 \cdot \bar{t}_j(x))$$
$$= v_j(x) - \bar{t}_j(x)$$
$$\leqq v_j(x) \leqq v_j({}_i x).$$

Hence, if ${}_i x \notin NE(t, r)$, then $\bar{t}_j({}_i x_j, \bar{x}_i) > 0$ and $U_i(({}_i x_j, \bar{x}_i), t, r) \geqq U_i(({}_i x_j, x_i'), t, r)$ for all x_i'. Moreover, $\bar{t}_j({}_i x_j, \bar{x}_i) > 0$ implies $U_j(({}_i x_j, \bar{x}_i), t, r) = \bar{u}_j \geqq U_j((x_j', \bar{x}_i), t, r)$ for all x_j'. That is, if ${}_i x \notin NE(t, r)$, then $({}_i x_j, \bar{x}_i) \in NE(t, r)$. Thus, $\{{}_i x, ({}_i x_j, \bar{x}_i)\} \cap NE(t, r) \neq \emptyset$, and for all x_i',

$$U_i(({}_i x_j, x_i'), t, r) = \begin{cases} v_i({}_i x_j, x_i') \leqq v_i({}_i x) \leqq \bar{u}_i & \text{if } \bar{t}_j({}_i x_j, x_i') = 0 \\ v_i({}_i x_j, x_i') + (v_j({}_i x_j, x_i') - \bar{u}_j) \leqq \bar{u}_i & \text{if } \bar{t}_j({}_i x_j, x_i') > 0 \end{cases}$$

in the subgame (2-3).

Thus, (1)-(2-4) constitutes a subgame perfect equilibrium where $\bar{t}$ and $\bar{r}$ are announced in the first stage and $\bar{x}$ is played in the second stage on the equilibrium path, and $U_i(\bar{x}, \bar{t}, \bar{r}) = \bar{u}_i$. ■

Remark 1. The point of the mechanism is that even if individual deviation from $(\bar{t}, \bar{r})$ occurs, any transfer is carried out in accordance with $\bar{t}_i$ of some i. If 1 is the deviator and $t_1 \neq \bar{t}_1$, then $r_2(t_1) = \bar{r}_2(t_1) = 0$ and no transfer from 1 to 2 takes place. Thus any individual deviation from $(\bar{t}, \bar{r})$ cannot promote transfer where $x_i \neq \bar{x}_i$ for all i and the transferer's, say i's, payoff after transfer is always equal to $\bar{u}_i$. Therefore, for any i's deviation in Stage 1, $\{{}_i x, ({}_i x_i, \bar{x}_j)\} \cap NE(t, r)$ and $\{{}_i x, ({}_i x_j, \bar{x}_i)\} \cap NE(t, r)$ are non-empty in the subgames (2-1) and (2-3) respectively, and i's payoff after transfer is no more than $\bar{u}_i$ in the equilibria.

Remark 2. It is a corollary of Proposition 1 that pure equilibrium strategies and outcomes of the underlying game are supportable.

Corollary 1. *If $x \in NE$, then $(x, v(x))$ is supportable.*

Note that $\bar{t}$ in the proof of Proposition 1 is sure to be feasible when $\bar{u}_i \geqq 0$ for all i. That is, even the following proposition holds in the model, which implies that any set of efficient actions maximizing the total payoff is feasibly supportable with some payoff distribution if there exists a pure equilibrium of the underlying game in which each player enjoys nonnegative payoff without side payments.

Proposition 2. *$(\bar{x}, \bar{u})$ such that $\bar{u}_i \geqq 0$ for all i and $\sum_i \bar{u}_i = \sum_i v_i(\bar{x}) \geqq \sum_i v_i(x)$ for all $x \in X$ is feasibly supportable if there exists ${}_i x$ for all i such that ${}_i x \in NE$ and $v_i({}_i x) \leqq \bar{u}_i$.*

4. Concluding Remarks

In line with Yamada (2003), we found a class of (feasible) simpler side contracts for two-player games which may induce play of efficient actions in equilibria if there is a pure Nash equilibrium in the underlying game. The result shows that the contracts proposed by Yamada (2003) may be more complicated than necessary to evoke efficient play. Are they really required to be interdependent? Even if the answer is yes, then, to what extent? Those are the problems that we should address in future research.

References

Coase, R. H. (1960) "The Problem of Social Cost" *Journal of Law and Economics* **3**, 1-44.

Jackson, M. O., and S. Wilkie (2005) "Endogenous Games and Mechanisms: Side Payments Among Players" *Review of Economic Studies* **72**, 543-566.

Yamada, A. (2003) "Efficient Equilibrium Side Contracts" *Economics Bulletin* **3**, No. 6, 1-7.

Yamada, A. (2005) "Alternate Contracts for Side Payments" *Economics Bulletin* **3**, No. 46, 1-5.