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Subshifts of Finite Range and Countable State Sofic Shifts

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1 Introduction

In previous paper [2], we succeeded to find a relation between the finite range structure condition (abrr. FRS) and countable state sofic shifts, that is, piecewise invertible systems with FRS define countable state sofic shifts. Such piecewise invertible systems with FRS give nice examples of countable state symbolic dynamics. In [2], we showed that the motion of piecewise invertible maps with FRS is described by labelled graphs with finitely many vertices and countably many edges. The key of finding such graphs is to note the following equivalence relation on the set of all semi-infinite admissible sequences:

$$(\ldots a_{-n}\ldots a_{-1}) \sim (\ldots b_{-n}\ldots b_{-1}) \Leftrightarrow \bigcap_{n=1}^{\infty} T^n X_{a_{-n}\cdots a_{-1}} = \bigcap_{n=1}^{\infty} T^n X_{b_{-n}\cdots b_{-1}}.$$

Under the FRS condition, the number of elements of the set $\mathcal{U}=\{T^n X_{a_1\cdots a_n}\colon \forall n>0, \forall X_{a_1\cdots a_n}\}$ is finite, so the limit $\bigcap_{n=1}^{\infty} T^n X_{a_{-n}\cdots a_{-1}}$ is exactly some $U\in\mathcal{U}$. This is a reason why we say that FRS implies countable state sofic shift. However, $\mathcal{U}$ can be a countable set in some class of piecewise invertible maps, and under some conditions we may obtain nice invariant measures. The key property for the existence of such nice invarinat measures may be considered as follows:

$\inf\{\lambda(U): U\in\mathcal{U}\}>0.$

Though we are still interested in case when $\mathcal{U}$ is a countable set, we will not discuss on such a case in this paper because the graph we will obtain has countably many vertices. For this reason, we can not expect any information from the labelled graph which determines the original system. Our main purpose in this paper is to study shifts obtained from labelled graphs G with finitely many vertices and countably many edges. In section 3, under the irreducibility and the right resolving property we will show that the factor map which is obtained by reading off the labels of the edges of G is finite to one and show the existence of a magic word. Combining these results, we have an analogy to the following well-known fact in finite state symbolic dynamics:

A finite-to-one factor map is one to one almost everywhere if and only if it has a *right resolving block (magic word).*

2 Notation and preliminaly results

Let T be a transformation on a bounded domain $X(\subset\mathbf{R}^n)$ and $Q=\{X_a\}_{a\in I}$ be a countable partition of X satisfying the following conditions:

1. Q is the generating partition (i.e., $\vee_{m=0}^{\infty}T^{-m}Q$ is the partition into points).

2. Each X_a is a connected subset of X with piecewise smooth boundary, and X_a can not intersect any X_b with $b\neq a$.

3. T is a locally homeomorphism on each X_a.

We call such T *a piecewise invertible map* and call the triple (T, X, Q) *a piecewise invertible system.* If *int* $(X_{a_1}\cap T^{-1}X_{a_2}\ldots\cap T^{-(n-1)}X_{a_n})\neq\emptyset$, then we say that the sequence $(a_1\ldots a_n)$ is *admissible* and denote $X_{a1}\cap$

$T^{-1}X_{a_2}\cap\ldots\cap T^{-(n-1)}X_{a_n}$ by $X_{a_1\cdots a_n}$. We call $X_{a_1\cdots a_n}$ *a cylinder of rank* n. $\mathscr{L}^n$ denotes the set of all cylinders of rank n. $|a_1\ldots a_n|$ stands for the length of the sequence $(a_1\ldots a_n)$. We denote $\mathscr{A}_n$ the set of all admissible sequences of length n and $\mathscr{A}=\bigcup_{n=1}^{\infty}\mathscr{A}_n$. Put $\mathscr{L}=\bigcup_{n=1}^{\infty}\mathscr{L}^n$. Let Σ^* be the set of all right semi-infinite sequences $(a_1a_2\ldots a_n\ldots)$ satisfying $(a_1\ldots a_n)\in\mathscr{A}$ for all $n>0$. For $(a_1a_2\ldots a_n\ldots)\in\mathscr{A}$, put $\rho(a_1a_2\ldots)=\bigcap_{i=0}^{\infty}T^{-i}X_{a_{i+1}}$. It follows from the genarator condition that $\rho(a_1a_2\ldots)$ is at most single point of X. Put $\Sigma_\emptyset=\{(a_1a_2\ldots)\in\Sigma^*: \rho(a_1a_2\ldots)=\emptyset\}$. Define

$$\Sigma^{*\prime}=\bigcap_{i=0}^{\infty}\sigma^{*-i}\{(a_1a_2\ldots)\in\Sigma^*\setminus\Sigma_\emptyset: \rho(a_1a_2\ldots)\in X_{a_1}\},$$

where σ^* is the left shift on Σ^*, i.e., $\sigma(a_1a_2\ldots)=(a_2a_3\ldots)$. We remark that Σ^* and $\Sigma^{*\prime}$ are σ^*-invariant.

Proposition 2.1 ρ: $\Sigma^{*\prime}\to X$ *is a bijective, continuous shift commuting map, i.e.,* $T\rho=\rho\sigma^*$.

We call the triple $(\Sigma^{*\prime},\sigma^*,\rho)$ a realization of $(T, X, Q, \mathcal{U})$ or a symbolic dynamics of $(T, X, Q, \mathcal{U})$,

We will define a version of the Markov property.

Definition We say that $X_{a_1\cdots a_n}\in\mathscr{L}$ is *a Markov cylinder* if for $\forall X_{b(m)}\in\mathscr{L}^m$ satisfying *int* $(T^mX_{b(m)}\cap X_{a(n)})\neq\emptyset$, *int* $X_{a(n)}\subset T^mX_{b(m)}$.

Let $\mathscr{M}^n$ be the set of all Markov cylinders of rank n and put $\mathscr{M}=\bigcup_{i=1}^{\infty}\mathscr{M}^n$. Assume that

(C-1) T is onto (i.e., $X=\bigcup_{a\in I}\mathrm{T}X_a$).

Then it follows from the condition 3 that T is a non-singular transformation with respect to Lebesgue measure λ of X. This fact and the local invertibility of T allows us to have the next result.

Proposition 2.2 *Let* (T, X, Q) *be a piecewise invertible system. For a Markov cylinder* $X_{a_1\cdots a_n}$, *the following conditions are equivalent.*

(a) $(b_1 \ldots b_l a_1 \ldots a_n) \in \mathscr{A}$.

(b) $T^{n+l} X_{b_1 \cdots b_l a_1 \cdots a_n} = T^n X_{a_1 \cdots a_n}$ (λ mod 0).

(c) $T^l X_{b_1 \cdots b_l} \supseteqq \mathrm{int}\ X_{a_1 \cdots a_n}$.

Definition We say that T has the *k-Markov property* if $\mathscr{M}^k = \mathscr{L}^k$

Denote $\mathcal{U}^k = \{T^k X_{a_1 \cdots a_k} : (a_1 \ldots a_k) \in \mathscr{A}_k\}$ and put $\mathcal{U} = \bigcup_{k=1}^{\infty} \mathcal{U}^k$.

Remark A $\mathcal{U} = \mathcal{U}^k \Leftrightarrow T$ has the k-Markov property. If T satisfies the k-Markov property, then we call a quadraple $(T, X, Q, \mathcal{U})$ a *piecewise invertible k-Markov system*. Its symbolic dynamics is exactly countable state 'k-Markov shift'([3]).

Definition Let (T, X, Q) be a piecewise invertible system such that $\mathcal{U}$ consists of only finitely many subsets of X with positive Lebesgue measure. Then we say that T guarantees a *finite range structure* (abrr.FRS) and we call the quadraple $(T, X, Q, \mathcal{U})$ *a piecewise invertible system with FRS*.

Remark B The subclass $\mathscr{M}$ of $\mathscr{L}$ has the "strong playback property", i. e., for $\forall X_{a(n)} \in \mathscr{M}$ and for $\forall X_{b(m)} \in \mathscr{L}$ such that $X_{b(m)a(n)} \in \mathscr{L}$, $X_{b(m)a(n)}$ also belongs to $\mathscr{M}$.

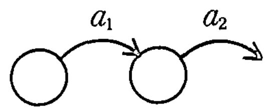
In fact, as the admissibility of the sequence $c(l) \cdot b(\mathrm{m}) a(\mathrm{n})$ implies the admissibility of $c(l) b(\mathrm{m}) \cdot a(\mathrm{n})$, the above assertioon is obtained immediately.

Remark C If $X_{a(n)}$ is a Markov cylinder, then for any $X_{b(m)} \in \mathscr{L}$ such that $X_{a(n)b(m)} \in \mathscr{L}$, $X_{a(n)b(m)}$ also belongs to $\mathscr{M}$.

3 Subshifts of finite range

Let G be a labelled graph with finitely many vertices and infinitely many edges, with the property:

(1) The follower set of the vertices are distinct. Here, a follower set $\mathscr{F}(v)$ of vertex v is by definition the set of infinite sequences $a_1 a_2 \ldots$



(Figure 1)

which label paths beginning at v. (Figure 1)

(2) G is irreducible, i.e., for any vertices v_1, v_2, there exists a path in G from v_1 to v_2.

(3) G is a right resolving graph, i.e., no two edges beginning at the same vertex have the same label.

Let Y be the space of all infinite sequences $y = \dots y_{-1} y_0 y_1 \dots$ which label path in G. Let Y_- be the set of all left semi-infinite sequences appearing in points of Y, and define for $(\dots a_{-n} \dots a_{-1}) \in Y_-$ $\mathscr{F}(\dots a_{-n} \dots a_{-1})$ is the set of right semi-infinite sequences $(a_0 a_1 \dots)$ such that $(\dots a_{-1})(a_0 \dots) \in Y$. For each $n > 0$ let $\mathscr{F}(a_{-n} \dots a_{-1})$ be the set of right semi-infinite sequences $(a_0 a_1 \dots)$ such that $(a_{-n} \dots a_{-1})(a_0 a_1 \dots)$ apperar in points of Y. Then,

$$\mathscr{F}(a_{-n} \dots a_{-1}) \supseteqq \mathscr{F}(a_{-(n+1)} a_{-n} \dots a_{-1}),$$

so

$$\lim_{n \to \infty} \mathscr{F}(a_{-n} \dots a_{-1}) = \bigcap_{n>0} \mathscr{F}(a_{-n} \dots a_{-1}) = \mathscr{F}(\dots a_{-n} \dots a_{-1}).$$

Define an equivalence relation over Y_- as follows:

$$(\dots a_{-n} \dots a_{-1}) \sim (\dots b_{-n} \dots b_{-1}) \Leftrightarrow \mathscr{F}(\dots a_{-n} \dots a_{-1}) = \mathscr{F}(\dots b_{-n} \dots b_{-1}).$$

Theorem 3.1 *The number of equivalence classes is only finite.*

Proof For each $(\dots a_{-n} \dots a_{-1}) \in Y$, we have a subset of vertex set $V = \{v_{i_1}, \dots v_{i_k}\}$ such that $\mathscr{F}(\dots a_{-n} \dots a_{-1}) = \bigcup_{v \in V} \mathscr{F}(v)$. As the number of vertices is only finite, $\{\mathscr{F}(\dots a_{-n} \dots a_{-1}) : (\dots a_{-n} \dots a_{-1}) \in Y_-\}$ is at most a finite set.

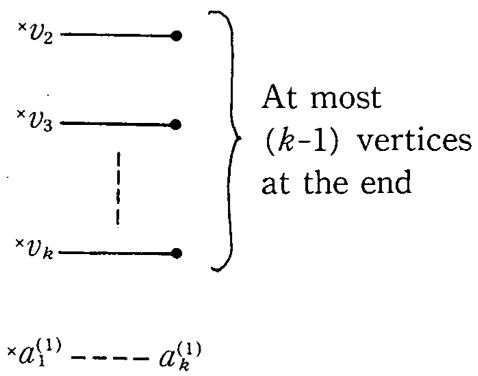
Remark D In the above sence, we say that subshifts which are obtained from labelled graphs satisfying (1),(2) and (3) are countable state sofic shifts. We call such a shift *subshift of finite range*.

By standard way, we obtain the Shannon cover which is a minimal right resolving graph and so that we have a finite to one factor map which defines the countable state sofic shift occurred from the graph G. So a subshift of finite range is a finite to one factor image of SFT which is obtained from its Shannon cover. Furthermore we can show the existence of a right resolving block.

Theorem 3.2 *Assume that (1), (2) and (3). Then there exists a word* $m=m_1...m_q$ *such that if* y *is an infinite sequence which labels an infinite path in* G, *and* $y_1...y_q=m$, *then for any infinite paths* x,x' *in* G *which are labelled by* y, *we have* $x_i=x'_i$ *for* $i>q$.

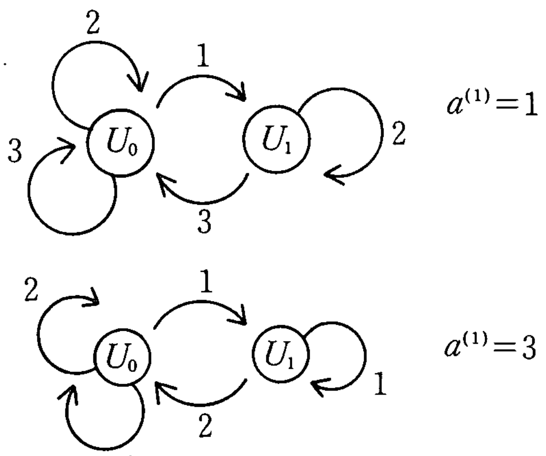
(We call such a word m, *a right resolving block or a magic word*.)

Proof Let $v_1,...v_k$ be the vertices in G. Without loss of generality, assume $k>1$. Pick a word $a^{(1)}=a_1^{(1)}...a_k^{(1)}$ such that $a^{(1)}$ labels a path from at least one of the vertices, but $a^{(1)}$ does not label a path from at least one of the vertices. Picture possible vertices

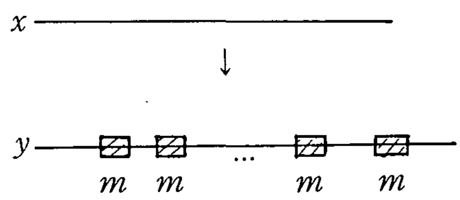


(Figure 2)

in G setting one corresponding position in $a^{(1)}$. For example assuming no path which is labelled by $a^{(1)}$ can begin at vertices v_1. (Figure 2). We remark that if for every $n>0$, $y_1 \ldots y_n$ labels a path beginning at a vertex v, then the sequence $y=y_1y_2\ldots$is in$\mathscr{F}(v)$. In fact, for each n, let $a_1^{(n)}a_2^{(n)}\ldots a_n^{(n)}$ be a path in G labelled by $y_1\ldots y_n$ such that initial vertex of y_1 is $v=v_1$. Then some vertex v_2 must occur as the initial vertex of $a_2^{(n)}$ for infinitely many n. For these n, we have $a_1^{(n)}$ an edge in G from v_1 to v_2 labelled by y_1. Choose a subsequence of the paths so that the initial edge $a_1^{(n)}$ is the same a_1. For a subsequence of this subsequence the second edge $a_2^{(n)}$ is the same a_2. And so on, $a_1a_2a_3\ldots$ This gives an infinite path labelled by y. By the resolving property, a path in G with a given initial vertex and a given labelling word is uniquely determined. So if the number of vertices in G is k, then at most $(k-1)$ vertices can be the terminal vertex of a path in G labelled by $a_1^{(1)}$. Consider the set E_1 of terminal vertices of such words. If $^{\|}E_1=1$, then done. If not, pick a word $a^{(2)}$ which labels a path from some vertex in E_1, but not from every vertex in E_1. Let E_2 be the set of terminal vertices of words labelled by



(Figure 3)



(Figure 4)

$a^{(1)}a^{(2)}=a_1^{(1)}\dots a_{k_1}^{(1)}a_1^{(2)}\dots a_{k_2}^{(2)}$.

Now $k > {}^{\|}E_1 > {}^{\|}E_2 > \dots > {}^{\|}E_j$. Continue until ${}^{\|}E_j=1$. The finiteness of number of vertices allows us to do it. Let $m=a^{(1)}$ $a^{(2)}\dots a^{(j-1)}$. (Figure 3).

We have further:

Corollary 3.1 *Suppose $y=\dots y_{-1}y_0y_1\dots$, m is the word produced by Theorem 3.2, y labels an infinite path in G, and we have G satisfying the assumptions (1), (2) and (3) and for infinitely many negative integers n, $y_{n+1}\dots y_{n+q}=m$. Then there is only one infinite path in G labelled by y.* (Figure 4).

Now we ask: when a finite to one factor map with a right resolving block is one to one almost everywhere? In case of finite state, we have usually an ergodic Borel invariant measure which attains a maximal entropy, so that we can show the following fact:

Proposition 3.1 *Again assume (1), (2), (3), and let m be the right resolving block as in Theorem 3.2. Let Y be the space of all infinite sequences $y=\dots y_{-1}y_0y_1\dots$ which label path in G, and have the usual topology (a basis for the topology is given by closed open sets of form $y_i\dots$ $y_j=w$.) Suppose μ is an ergodic Borel probability measure on Y such that for every non-empty open set O in Y, $\mu(O)>0$. (This means that for those sets of form $y_i\dots y_j=w$, the μ-measure is posotive.) Then the set*

$\{y \in Y$: $m = y_{n+1} \dots y_{n+q}$ for infinitely many positive and for infinitely many negative integers $n\}$ has μ-measure 1.

Proof Let $O = \{y \in Y\colon y_1 \dots y_q = m\}$, and $\mu(O) > 0$. It follows from the ergodic theorem that

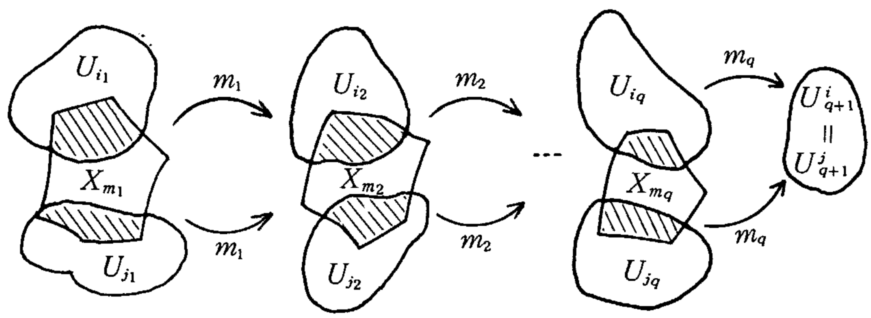
$$\mu\{y \in Y: \lim_{n \to \infty} 1/n \sum_{j=0}^{n-1} 1_m(\sigma^j y) = \mu(O)\} = 1.$$

For such $y \in Y$, $1_m(\sigma^j y) \neq 0$ for infinitely many $j > 0$. It is not difficult to show that $1_m(\sigma^j y) \neq 0$ for infinitely many $j < 0$. Intersection of two measure 1 sets has measure 1.

In case of countable state, we can not obtain such an invariant measure usually. We have a sufficient condition for the existence of an ergodic finite invariant measure equivalent to Lebesgue measure for piecewise invertible maps with FRS in [1]. Even if we do not have such nice invariant measure, the following condition which is given in [3] is enough to answer to our question.

(C-2) $\bigcup_{U \in \mathcal{U}_M} U = X(\lambda \ mod\ 0)$.

Theorem 3.3 *Let $(T, X, Q = \{X_a\}_{a \in I}, \mathcal{U})$ be a piecewise invertible system with FRS satisfying $X \in \mathcal{U}$. Let $\pi\colon \sigma \to \sigma'$ be the one-block map obtained by Theorem 4.1 in [3]. Assume that (C-2) is valid. Then π*



$T\ (U_{i_q} \cap X_{m_q}) = T\ (U_{j_q} \cap X_{m_q}) = T^q X_{m_1 \dots m_q}$

(Figure 5)

is one to one almost everywhere.

Lemma 3.1 *Markov cylinders give right resolving blocks (magic words).*

Proof It follows from the definition of Markov cylinder that if $x_1 \dots x_q$, $x'_1 \dots x'_q$ are preimages of $m = m_1 \dots m_q$, then we have $x_{q+1} = x'_{q+1}$, immediately. In fact we have the following situation: (Figure 5).

Proof of Theorem 3.3 As mentioned before, π is a finite to one factor map. First we noto that:

$\forall x \in X, \exists U \in \mathcal{U}_M$ such that $U \ni x$, so that $\exists X_{m_1 \cdots m_q} \in \mathscr{M}$ satisfying $m_1 \dots m_q a_0(x) \dots a_n(x) \in \mathscr{A} (\forall n > 0)$.

From Proposition 3.3 in [3] and Lemma 3.2, we can apply Corollary 3.1 to our situation in order to obtain the desired result.

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